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Symplectic geometry of $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ algebras*

Shao-Ming Fei† and Han-Ying Guo‡

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China and Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China

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Abstract. We give a complete description of q-deformation of SU(2) algebra in terms of symplectic geometry. The geometric meanings of such a deformation are manifestly shown at classical as well as at quantum level. As a model we study the classical and quantum dynamics of a particle with q-spin moving in electromagnetic fields in detail.

1. Introduction

The q-deformation, or the 'quantum group' of SU(2) algebra, $SU_q(2)$, and its representations have been investigated by many authors [1, 2]. It is usual to think that the q-deformations always emerge together with quantization of the systems characterized by Planck's constant $\hbar$§. In the classical limit $\hbar\to 0$, the q-deformed algebraic relations reduce to the usual Lie algebraic relations. Very recently, however, one of us (HYG) and his collaborators have pointed out that this is not the case in principle [3]. By means of the classical harmonic oscillators, they realized the q-deformed SU(2) algebra at classical level, $SU_{q,\hbar\to 0}(2)$, as was denoted in [3], in the sense of classical Poisson brackets of deformed observables $J'_\pm$ and J'_3 based upon the undeformed phase space with undeformed symplectic form. After taking canonical quantization of the harmonic oscillators, they recovered the quantum q-deformed SU(2) algebra, $SU_{q,\hbar}(2)$, whose generators, i.e. the quantum deformed observables $J'_\pm$, J'_3, are the same as the ones proposed in [4–7] based upon the q-deformed quantum harmonic oscillators.

In this paper, we will observe this interesting discovery from a different angle. That is, instead of a concrete realization system such as the harmonic oscillators, we deal with the problem abstractly in the framework of the symplectic geometry of the intrinsic angular momentum space [8]. We find that the classical q-deformation of SU(2) algebra, $SU_{q,\hbar\to 0}(2)$, can be realized by the deformed Poisson brackets given by the deformed symplectic form together with deformed observables and after geometric quantization based upon the deformed symplectic geometry, the quantum q-deformation SU(2) algebra, $SU_{q,\hbar}(2)$, can be reached as well. In other words, we find a complete and systematic description of such an algebraic q-deformation at both classical and quantum levels based upon the symplectic geometry and its q-deformed version.

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† Address for correspondence: Physics Department, Zhejiang University, Hangzhou 310027, People's Republic of China

‡ Address for correspondence: Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China.

§ In certain cases, by some parameter playing a role loosely like $\hbar$.

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One of the advantages of the approach presented in this paper is that we may apply this q-deformed symplectic form for the intrinsic angular momentum to various cases. As an example, we study a model of a changed particle with q-deformed spin, which we call q-spin, moving in given external magnetic fields in some detail.

The paper is organized as follows. In section 2, we study how to realize the classical q-deformation of SU(2) algebra, $\mathrm{SU}_{q,\hbar\to 0}(2)$, by means of the deformation of the symplectic geometry of the intrinsic angular momentum space rather than by means of the deformed observables on undeformed phase space [3]. In section 3, we deal with the prequantization and polarization of the system based upon the deformed symplectic geometry and show the quantum q-deformed algebra $\mathrm{SU}_{q,\hbar}(2)$. The model of a charged particle with q-spin moving in given magnetic fields is discussed in sections 3 and 4. Finally, a few concluding remarks are given in section 5.

2. Symplectic geometry of $\mathrm{SU}_{q,\hbar\to 0}(2)$ algebra

The spin, as an intrinsic angular momentum, has a classical representation in the version of symplectic geometry [8]. For the usual spin vector $S_0=(S_{01}, S_{02}, S_{03})$ such that

$$S_{01}^2+S_{02}^2+S_{03}^2=S_0^2 \tag{1}$$

where S_0 is a given constant, there corresponds to a symplectic form

$$\omega_0=-\frac{1}{S_0^2}\sum_{ijk=1}^{3}\varepsilon_{ijk}S_{0i}\,\mathrm{d}S_{0j}\wedge\mathrm{d}S_{0k} \qquad i,j,k=1,2,3 \tag{2}$$

and the Lagrange bracket for a free particle with spin S_0 is given by the symplectic form

$$\Omega_0=\sum_i \mathrm{d}p_i\wedge\mathrm{d}q^i+\omega_0.$$

On the one-parameter group of rotations around the ith axis there exist Hamiltonian vector fields $X_{S_{0i}}$ with respect to ω_0

$$X_{S_{0i}}=\sum_{jk}\varepsilon_{ijk}S_{0j}\frac{\partial}{\partial S_{0k}}. \tag{3}$$

By using the relation (1) it is easy to show that the following formulae are satisfied

$$X_{S_{0i}}\rfloor\,\omega_0=-\mathrm{d}S_{0i}$$

$$\omega_0(X_{S_{0i}}, X_{S_{0j}})=[S_{0i}, S_{0j}]_{\mathrm{PB}}$$

and

$$[S_{0i}, S_{0j}]_{\mathrm{PB}}=-X_{S_{0i}}S_{0j}=\varepsilon_{ijk}S_{0k} \tag{4}$$

where $X_{S_{0i}}\rfloor\,\omega_0$ denotes the left inner product of $X_{S_{0i}}$ and ω. Formula (4) is just the Poisson bracket of spin components.

Now similar to what is discussed on the spin sphere $S_{01}^2+S_{02}^2+S_{03}^2=S_0^2$, let us consider the case on a q-deformed sphere, i.e. a q-spin sphere defined by

$$S_1^2+S_2^2+\frac{(\sinh\gamma S_3)^2}{\gamma\sinh\gamma}=S_\gamma^2 \tag{5}$$

which is 'flatter' than the spin sphere (1), where S_γ is a given constant and $\gamma=\ln q$, the deformation parameter taking values from zero to infinite. It is obvious that the q-spin sphere becomes the original spin sphere when γ approaches zero.

In this case the symplectic form for a free particle with q-sin $\boldsymbol{S}$ is

$$\Omega_\gamma=\sum_i \mathrm{d}p_i\wedge \mathrm{d}q^i+\omega_\gamma$$

where ω_γ is the symplectic form for the q-spin part

$$\omega_\gamma=-\frac{1}{S^2}\left(S_1\,\mathrm{d}S_2\wedge\mathrm{d}S_3+S_2\,\mathrm{d}S_3\wedge\mathrm{d}S_1+\frac{\tanh\gamma S_3}{\gamma}\mathrm{d}S_1\wedge\mathrm{d}S_2\right). \tag{6}$$

The Hamiltonian vector fields of S_i with respect to ω_γ now take the forms

$$\begin{aligned}X_{S_1}&=S_2\frac{\partial}{\partial S_2}-\frac{\sinh 2\gamma S_3}{2\sinh\gamma}\frac{\partial}{\partial S_2}\\ X_{S_2}&=\frac{\sinh 2\gamma S_3}{2\sinh\gamma}\frac{\partial}{\partial S_1}-S_1\frac{\partial}{\partial S_2}\\ X_{S_3}&=S_1\frac{\partial}{\partial S_2}-S_2\frac{\partial}{\partial S_1}.\end{aligned} \tag{7}$$

It should be noticed that the symplectic form ω_γ and the corresponding Hamiltonian vectors, according to the constraint of q-spin sphere (5), are subjected to the relations

$$\begin{aligned}&X_{S_i}\lrcorner\,\omega=-\mathrm{d}S_i\\ &[X_{S_i},X_{S_j}]=-X_{[S_i,S_j]_{\mathrm{PB}}}\\ &\omega(X_{S_i},X_{S_j})=[S_i,S_j]_{\mathrm{PB}}\end{aligned}$$

which uniquely give rise to the Poisson brackets

$$\begin{aligned}&[S_1,S_2]_{\mathrm{PB}}=\frac{\sinh 2\gamma S_3}{2\sinh\gamma}\\ &[S_1,S_3]_{\mathrm{PB}}=-S_2\\ &[S_2,S_3]_{\mathrm{PB}}=S_1.\end{aligned} \tag{8}$$

Formulae (8) are just the q-deformed algebras of SU(2), that is

$$\begin{aligned}&[S_+,S_-]_{\mathrm{PB}}=-\mathrm{i}\frac{\sinh 2\gamma S_3}{\sinh\gamma}\\ &[S_3,S_\pm]_{\mathrm{PB}}=\mp\mathrm{i}S_\pm\end{aligned} \tag{9}$$

where $S_\pm=S_1\pm\mathrm{i}S_2$. However it is realized at classical level in the sense that the Poisson brackets are defined by the q-deformed symplectic form ω_γ in (6). It should be mentioned again that the approach to the classical q-deformed SU(2) algebra, denoted by $\mathrm{SU}_{q,\hbar\to0}(2)$ as in [3], presented in this paper is different from the one in [3] where the observables $J'_\pm$, J'_3 are deformed and form $\mathrm{SU}_{q,\hbar\to0}(2)$ algebra in the sense of Poisson brackets defined by the undeformed symplectic form, although the resultant q-deformed algebraic relations are the same. It is also notable that after deformation, $\gamma>0$, a rotation symmetry around the third component is still preserved, i.e. there still exists a U(1) symmetry, although the SU(2) symmetry is broken.

3. Particle with q-spin and quantum q-deformed algebra $\mathrm{SU}_{q,\hbar}(2)$

In this section we discuss the symplectic geometry of non-relativistic charged particles with q-spins in given external electromagnetic fields and see a set of constraints, which lead to the quantum q-deformed algebra $\mathrm{SU}_{q,\hbar}(2)$, and which will be imposed on the q-spin value after quantization.

The classical state of a non-relativistic particle with usual spin $S_0>0$ is specified by the position $\boldsymbol{q}$, the momentum $\boldsymbol{p}$ and the spin vector $\boldsymbol{s}_0$. Hence the phase space under consideration is

$$X_0=R^3\times R^3\times S_0^2$$

where S_0^2 denotes the sphere in R^3 of radius S_0. In a given external magnetic field $\boldsymbol{B}$ the energy in classical state $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{S}_0)$ is given by

$$H(\boldsymbol{P}, \boldsymbol{q}, \boldsymbol{S}_0)=\frac{\boldsymbol{P}^2}{2m}+eV(\boldsymbol{q})-\frac{e}{m}\boldsymbol{S}_0\cdot\boldsymbol{B}(\boldsymbol{q}) \tag{10}$$

where e is the electric charge of the particle, $\boldsymbol{B}(\boldsymbol{q})$ the magnetic field and $V(\boldsymbol{q})$ the electric potential. The corresponding symplectic form is

$$\Omega_0=\sum_{i=1}^{3}\mathrm{d}p_i\wedge\mathrm{d}q^i+\frac{1}{2}e\sum_{ijk}c_{ijk}B_i\,\mathrm{d}q^i\wedge\mathrm{d}q^k+\omega_0 \tag{11}$$

where ω_0 is given by (2).

For a non-relativistic particle with q-spin, its phase space becomes $X=R^3\times R^3\times S_\gamma^2$. Here S_γ^2 denotes the q-spin defined by (15). The Hamiltonian and symplectic forms now appear as

$$H=\frac{p^2}{2m}+eV(\boldsymbol{q})-\frac{e}{m}\boldsymbol{S}\cdot\boldsymbol{B}(\boldsymbol{q}) \tag{12}$$

$$\Omega=\sum_i\mathrm{d}p_i\wedge\mathrm{d}q^i+\tfrac{1}{2}e\sum_{ijk}\varepsilon_{ijk}B_i\,\mathrm{d}q^i\wedge\mathrm{d}q^k+\omega_\gamma \tag{13}$$

where ω_γ is the q-deformed symplectic form (6).

Similar to the case in $\boldsymbol{S}_0$, let L be the prequantization line bundle of (X,Ω). The prequantization condition [8] is satisfied if and only if the de Rham cohomology class $[-h^{-1}\Omega]$ of $-h^{-1}\Omega$ is integrable, where h is Planck's constant. Since $\boldsymbol{B}$ is globally defined and div $\boldsymbol{B}=0$, the second term on the right-hand side of (13) is an exact form. Therefore

$$[-h^{-1}\Omega]=[-h^{-1}\omega_\gamma].$$

Integrating the right-hand side of (13) over the q-spin sphere $\{0\}\times\{0\}\times S_\gamma^2\subseteq X$, we have

$$\int_{S_\gamma}\omega_\gamma=-\frac{1}{S^2}\left[2V+\pi\left(S_\gamma^2\frac{\tanh\gamma S_3}{\gamma}-\frac{\gamma S_3-\tanh\gamma S_3}{\gamma^2\sinh\gamma}\right)\right]\Bigg|_{\sinh\gamma S_3/\sqrt{\gamma\sinh\gamma}=-R}^{\sinh\gamma S_3/\sqrt{\gamma\sinh\gamma}=+R}$$

where V is the volume of the q-spin sphere,

$$\begin{aligned}V&=\int_{S_\gamma}\mathrm{d}S_1\,\mathrm{d}S_2\,\mathrm{d}S_3\\&=\left(\pi S_\gamma^2S_3-\frac{\pi}{2}\,\frac{\sinh 2\gamma S_3-2\gamma S_3}{2\gamma^2\sinh\gamma}\right)\Bigg|_{\sinh\gamma S_3/\sqrt{\gamma\sinh\gamma}=-R}^{\sinh\gamma S_3/\sqrt{\gamma\sinh\gamma}=+R}.\end{aligned}$$

Therefore

$$\int_{S_\gamma^2} \omega_\gamma = -4\pi \frac{\sinh^{-1}(\sqrt{\gamma \sinh \gamma} S_\gamma)}{\gamma}.$$

Setting

$$\frac{\sinh^{-1}(\sqrt{\gamma \sinh \gamma} S_\gamma)}{\gamma} = j\hbar$$

we get

$$-h^{-1} \int_{S_\gamma^2} \omega_\gamma = 4\pi h^{-1} j\hbar = 2j \tag{14}$$

which must be an integer if $[-h^{-1}\omega_\gamma]$ is integrable. Equation (14) implies the first Chern number of the bundle L, which should be an integer, i.e. $2j \in Z$ and hence j is an integer or half integer. It is clear that the total q-spin S_γ now takes some special values according to j,

$$S_\gamma = \frac{\sinh \gamma j}{\sqrt{\gamma \sinh \gamma}}. \tag{15}$$

Formula (15) is just the eigenvalues of the Casimir operator of quantum algebra $SU_q(2)$ [1] up to a constant factor $(\sinh \gamma/\gamma)^{1/2}$ that can be removed by redefining the generators S_1, S_2, S_3.

It is worthwhile to note that using equations (1)–(5), if we apply a similar deformation to S_0 as that to S_3, we have a q-spin sphere

$$S_1^2 + S_2^2 + \frac{(\sinh \gamma S_3)^2}{\gamma \sinh \gamma} = \frac{(\sinh \gamma S_\gamma)^2}{\gamma \sinh \gamma}$$

then from (15) the prequantization condition implies $S_\gamma = j\hbar$.

In order to obtain certain polarization to fulfil the geometric quantization of the system we may define two open sets U_+ and U_- in X by

$$U_\pm = \left\{ x \in X \,|\, S_\gamma \pm \frac{\sinh \gamma S_3}{\sqrt{\gamma \sinh \gamma}} \neq 0 \right\}$$

and introduce complex coordinates

$$Z_\pm = \frac{S_1 \mp \mathrm{i} S_2}{S_\gamma \pm \sinh \gamma S_3/\sqrt{\gamma \sinh \gamma}}. \tag{16}$$

Then the q-deformed symplectic form ω_γ becomes

$$\omega_\gamma = -2\mathrm{i} S_\gamma \left(S_\gamma^2 \frac{(1 - Z_\pm \overline{Z_\pm})^2}{(1 + Z_\pm \overline{Z_\pm})^2} \gamma^2 + \frac{\gamma}{\sinh \gamma} \right)^{1/2} \frac{\mathrm{d}\overline{Z_\pm} \wedge \mathrm{d}Z_\pm}{(1 + Z_\pm \overline{Z_\pm})^2}.$$

After the polarization being chosen such that the Hilbert space corresponding to q-spin is the holomorphic function of $Z_\pm$, we may get the quantum expressions $\hat{S}_i$ of operators S_i in terms of $Z_\pm$, $\bar{Z}_\pm$ and their derivatives, which give rise to, as in the case of canonical quantization, the commutators of the quantum q-deformed algebra of SU(2), $SU_{q,\hbar}(2)$, as follows

$$\begin{aligned} [\hat{S}_+, \hat{S}_-] &= [2\hat{S}_3] = \frac{\sinh 2\gamma \hat{S}_3}{\sinh \gamma} \\ [\hat{S}_3, \hat{S}_\pm] &= \pm \hat{S}_\pm . \end{aligned} \tag{17}$$

As the geometric meaning of q-deformation has been displayed explicitly at the classical and quantum levels, we will not investigate the quantum Hilbert space and operator expression in the framework of geometric quantization further. Rather, we will use the results of canonical quantization directly in the next section.

4. The model of a particle with q-spin moving in magnetic fields

The motion of a charged particle with usual spin S_0 in an external electromagnetic field is described by the Hamiltonian (10) and the symplectic form (11). It is easy to find that, as is well known, the spin vector $\boldsymbol{S}_0$ rotates in the ith direction when $B_i=$ constant and other components of $\boldsymbol{B}$ vanish. Such a motion may be described by triangular functions. However for a particle with q-spin, that is its spin components satisfy (5), things are rather different.

For a particle with q-spin, the Hamiltonian vector field of H with respect to the symplectic form (13) is given by

$$\begin{aligned} X_H = m^{-1}\sum_i p_i\frac{\partial}{\partial q^i}-\frac{e}{m_e}\bigg[&\left(B_2\frac{\sinh 2\gamma S_3}{2\sinh\gamma}-B_3S_2\right)\frac{\partial}{\partial S_1} \\ &+\left(B_3S_1+B_1\frac{\sinh 2\gamma S_3}{2\sinh\gamma}\right)\frac{\partial}{\partial S_2}+(B_1S_2-B_2S_1)\frac{\partial}{\partial S_3}\bigg] \\ &+e\sum_i\left[m^{-1}\sum_{jk}\varepsilon_{ijk}p_jB_k+\frac{\partial V}{\partial q^i}-(mc)^{-1}\sum_j S_j\frac{\partial B_j}{\partial q^i}\right]\frac{\partial}{\partial p_i} \end{aligned} \tag{18}$$

which satisfies

$$X_H \lrcorner\, \Omega = -\mathrm{d}H$$

$$[X_H, X_{S_i}]_{\mathrm{PB}} = X_{[H,S_i]_{\mathrm{PB}}}.$$

By using formula (18) it is easy to obtain the time-evolution equations of S_i,

$$\dot{S}_i=\frac{\mathrm{d}S_i}{\mathrm{d}t}=[S_i, H]_{\mathrm{PB}}=X_HS_i$$

i.e.

$$\begin{aligned} \dot{S}_3&=-\frac{e}{mc}(B_1S_2-B_2S_1) \\ \dot{S}_1&=-\frac{e}{mc}\left(B_2\frac{\sinh 2\gamma S_3}{2\sinh\gamma}-B_3S_2\right) \\ \dot{S}_2&=-\frac{e}{mc}\left(B_3S_1-B_1\frac{\sinh 2\gamma S_3}{2\sinh\gamma}\right) \end{aligned} \tag{19}$$

and also

$$\begin{aligned} \dot{p}_i&=e\left[m^{-1}\sum_{jk}\varepsilon_{ijk}p_jB_k+\frac{\partial V}{\partial q^i}-(mc)^{-1}\sum_j S_j\frac{\partial B_j}{\partial q^i}\right] \\ \dot{q}_i&=m^{-1}p_i. \end{aligned} \tag{20}$$

Let us focus on equation (19) in several special cases. First, we take $B_1=B_2=0$, $B_3=$ constant. Equation (19) becomes

$$\dot{S}_3=0 \qquad \dot{S}_1=\frac{e}{mc}B_3S_2 \qquad \dot{S}_2=-\frac{e}{mc}B_3S_1$$

The solutions are

$$S_1=A\sin(\omega_3t+\varphi)$$

$$S_2=A\cos(\omega_3t+\varphi)$$

$$S_3=A'=\text{constant}$$

where $\omega_3=eB_2/mc$, A and φ are integral constants to be determined by initial conditions. That is the same as the situation of spin S_0 except that A and A' should now satisfy the equation

$$A^2+\frac{(\sinh\gamma A')^2}{\gamma\sinh\gamma}=S_\gamma^2. \tag{21}$$

For the case of $B_1=$ constant and $B_2=B_3=0$, we have

$$\dot{S}_3=-\frac{e}{mc}B_1S_2 \qquad \dot{S}_1=0 \qquad \dot{S}_2=\frac{e}{mc}B_1\frac{\sinh 2\gamma S_3}{2\sinh\gamma}.$$

With the help of (5), it is easy to find that

$$S_1=\text{constant} \tag{22}$$

$$\dot{S}_3=-\omega_1^2\frac{\sinh 2\gamma S_3}{2\sinh\gamma} \tag{23a}$$

$$\dot{S}_2=aS_2+bS_2^2 \tag{23b}$$

where $\omega_1=eB_1/mc$, $a=\omega_1^2[(\gamma/\sinh\gamma)+2\gamma^2(S_\gamma^2-S_1^2)]$ and $b=2\gamma^2\omega_1^2$. Integrating (23a) once we have

$$\left(\frac{\mathrm{d}S_3}{\mathrm{d}t}\right)^2=-\omega_1^2\frac{(\sinh 2\gamma S_3)^2}{\gamma\sinh\gamma}+C_1^2\omega_1^2.$$

Hence

$$t=\int \mathrm{d}S_3\left(-\omega_1^2\frac{(\sinh\gamma S_3)^2}{\gamma\sinh\gamma}+C_1^2\omega_1^2\right)^{-1/2}-\frac{C_2}{\omega_1} \tag{24}$$

where C_1 and C_2 are integral constants. The integral in (24) can be transformed into standard elliptic integral if we set $\sinh\gamma S_3=\sqrt{C_1^2\gamma\sinh\gamma}\,\xi$ and $k^2=C_1^2\gamma\sinh\gamma/(1+C_1^2\gamma\sinh\gamma)$,

$$t=\int\frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(k'^2+k^2\xi^2)}}\frac{k'}{\omega_1}\sqrt{\frac{\sinh\gamma}{\gamma}}-\frac{C_2}{\omega_1} \qquad k'=\sqrt{1-k^2}.$$

Therefore ξ may be expressed by the elliptic function

$$\xi=\mathrm{cn}\left[\sqrt{\frac{\gamma}{\sinh\gamma}}\frac{1}{k'}(\omega_1t+C_2)\right]$$

and

$$\frac{\sinh \gamma S_3}{\sqrt{\gamma \sinh \gamma}} = C_1 \mathrm{cn}\left[\sqrt{\frac{\gamma}{\sinh \gamma}}\frac{1}{k'}(\omega_1 t + C_2)\right]. \tag{25}$$

When $\gamma \to 0$, the solution (25) becomes

$$S_2 = C_1 \cos(\omega_1 t + C_2) \tag{26}$$

Similarly for (23*b*), we have

$$t = \int \mathrm{d}S_2 \left(\frac{b}{2}\right)^{-1/2}\left(S_2^4 - \frac{2a}{b}S_2^2 + \frac{2}{b}C_3\right)^{-1/2} - \frac{C_4}{\omega_1} \stackrel{\mathrm{d}}{=} \int \mathrm{d}S_2 \left(\frac{b}{2}\right)^{-1/2}(P(S_2))^{-1/2} - \frac{C_4}{\omega_1}$$

with C_3 and C_4 as integral constants. The equation $P(S_2) = 0$ has the four solutions

$$\alpha_1 = \tfrac{1}{2}(k_1 + k_2) \qquad \beta_1 = \tfrac{1}{2}(k_1 - k_2)$$

$$\alpha_2 = \tfrac{1}{2}(k_2 - k_1) \qquad \beta_2 = \tfrac{1}{2}(-k_1 - k_2)$$

where

$$k_1 = \left(\frac{2a}{b} + 2\sqrt{\frac{2C_3}{b}}\right)^{1/2} \qquad k_2 = \left(\frac{2a}{b} - 2\sqrt{\frac{2C_3}{b}}\right)^{1/2}$$

and $\alpha_1 > \beta_1 > \alpha_2 > \beta_2$. Let

$$\frac{S_2 - \alpha_2}{S_2 - \beta_2} = \frac{\beta_1 - \alpha_2}{\beta_1 - \beta_2}\xi^2 \qquad k^2 = \frac{(\alpha_1 - \beta_2)(\beta_1 - \alpha_2)}{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}.$$

Then

$$t = \frac{2}{\sqrt{\frac{1}{2}b}\,k_1}\int \frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} - \frac{C_4}{\omega_1}.$$

Hence

$$\begin{aligned} \xi &= \mathrm{sn}\left[\frac{1}{2}\sqrt{\frac{b}{2}}\,k_1\left(t + \frac{t}{\omega_1}\right)\right] \\ S_2 &= \frac{1}{2}\,\frac{k_1(k_2 - k_1) + (k_1^2 - k_2^2)\xi^2}{k_1 - (k_1 - k_2)\xi^2}. \end{aligned} \tag{27}$$

Here S_1, C_1, C_2, C_3 and C_4 are subjected to the constraint (5).

Instead of triangular functions as in the case of S_0, we have seen that the motion is now described by elliptic functions sn and cn. When C_1, C_2, C_3 and C_4 are properly chosen the solutions of the motion become the ordinary ones at $\gamma \to 0$ by using the properties of sn and cn functions.

Let us now consider the quantum dynamics of the particle with q-spin. In the Schrödinger representation, the quantum operator of Hamiltonian H can be expressed as

$$\hat{H} = -\frac{e}{mc}\sum_j B_j \hat{S}_j - \frac{\hbar^2}{2m}\sum_{jk}\left(\frac{\partial}{\partial q^j} - i\frac{e}{\hbar}A_j\right)\left(\frac{\partial}{\partial q^k} - i\frac{e}{\hbar}A_k\right) + eV$$

where $\boldsymbol{A}$ is the magnetic potential vector, V the electric potential.

As exposed to the ordinary situation, the contribution from the spin is now changed to be $(e/mc)\,\Sigma_j\, B_j\hat{S}_j$. Here $\hat{S}_j$ have q-deformed eigenvalues and eigenstates as follows [1, 9]

$$\hat{S}_3|j\,m\rangle = m|j\,m\rangle$$

$$\hat{S}_\pm|j\,m\rangle = ([j\mp m][j\pm m+1])^{1/2}|j\,m\pm 1\rangle$$

in which $[x]=\sinh\gamma x/\sinh\gamma$ and $+j\leqslant m\leqslant j$. Having these q-deformed eigenvalue equations, we may solve the stationary Schrödinger equation. Clearly either when $B_1=B_2=0$ and $B_3=\text{constant}$ or for the case $\gamma\to 0$, the eigenvalues of the Hamiltonian are the same as the usual ones obtained in the case of the particle with undeformed spin moving in the magnetic fields.

5. Remarks

We have manifestly shown the geometric origin of the q-deformation of SU(2) both at classical and quantum levels and discussed the effects of such a deformation on the dynamics of simple physical systems.

It is of interest to notice that the deformation, defined by the transformation from spin sphere (1) to q-spin sphere (5), is closely related to quasiconformal transformations between these two Riemann surfaces. In fact, equation (16) may be regarded as the quasiconformal transformations between the undeformed set of coordinates and the deformed set if we eliminate S_3 with the help of (5). The situation, which is very similar to that which has been found in [3], may be studied further based upon the theory of quasiconformal transformations [10].

It should also be pointed out that (5) is only one kind of deformation of the spin sphere. Other deformations may also be taken into account, say the deformations might occur along more than one direction and so on. And various deformed algebras could be obtained with respect to different deformations. What is more, the discussions above may be extended to the q-deformations of other semisimple groups. As the Casimir operator of SU(2) defines a spin sphere, certain Casimir operators of other semisimple groups also determine similar intrinsic manifolds. The deformations of these intrinsic manifolds could give rise to the deformations of their algebras.

Finally, we would make some remarks on the deformation parameter $\gamma=\log q$. In this paper we have taken the deformation parameter γ to be real. It is notable when γ is the roots of unit. If $\gamma=i\gamma'$ and γ' is a real number, then the q-spin sphere becomes

$$S_1^2+S_2^2+\frac{(\sin\gamma' S_3)^2}{\gamma'\sin\gamma'}=S_\gamma^2. \tag{28}$$

It is of interest to see the roles of zeros in the triangular function. One of the consequences is that the commutation relation about 'spin' component S_1 and S_2 may be broken at certain points of S_3 or γ'. Since in this case $[S_1, S_2]=\mathrm{i}(\sin 2\gamma' S_3/2\sin\gamma')$, when $S_3=n\pi/\gamma'$ and n is an integer, one immediately gets $[S_1, S_2]=0$. This indicates that at those points the structure of the classical phase space is broken and consequently both S_1 and S_2 can be measured simultaneously if the original quantum theory is still used at those points. Hence, the deformations such as (28) not only remarkably change the algebraic properties of SU(2) but also may raise some problems in both classical and quantum theory. All these and relevant subjects are under investigation.

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